Ergodization time for linear flows on tori via geometry of numbers

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Abstract

In this paper, we give a new, short, simple and geometric proof of the optimal ergodization time for linear flows on tori. This result was first obtained by Bourgain, Golse and Wennberg in [BGW98] using Fourier analysis. Our proof uses geometry of numbers: it follows trivially from a Diophantine duality that was established by the author and Fischler in [BF13].

Let $n \geq 2$ be an integer, $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, $\alpha \in \mathbb{R}^n \setminus \{0\}$ and consider the linear flow on $\mathbb{T}^n$ defined by

$$X^t_\alpha(\theta) = \theta + t\alpha, \quad t \in \mathbb{R}, \quad \theta \in \mathbb{T}^n.$$

It is just the flow of the constant vector field $X_\alpha = \alpha$ on $\mathbb{T}^n$. Such flows play an important role in Hamiltonian systems, and their dynamical properties depend on the Diophantine properties of the vector α, as we will recall now.

Let us say that a vector subspace of $\mathbb{R}^n$ is rational if it admits a basis of vectors with rational components. We define F_α to be the smallest rational subspace of $\mathbb{R}^n$ containing α, so that $\Lambda_\alpha := F_\alpha \cap \mathbb{Z}^n$ is a lattice in F_α. In the special case where $F_\alpha = \mathbb{R}^n$, we have $\Lambda_\alpha = \mathbb{Z}^n$ and the vector α is said to be non-resonant: it is an elementary fact that the flow X^t_α is minimal (all orbits are dense) and in fact uniquely ergodic (all orbits are uniformly distributed with respect to Haar measure). In the general case where F_α has dimension d with $1 \leq d \leq n$, choosing a complementary subspace E_α of F_α, the affine foliation

$$\mathbb{R}^n = \bigsqcup_{v \in E_\alpha} v + F_\alpha$$

induces a foliation on $\mathbb{T}^n$ such that each leaf, which is just a translate of the d-dimensional torus $\mathcal{T}^d_\alpha := F_\alpha/\Lambda_\alpha$, is invariant by the flow and the restriction of the latter is minimal and uniquely ergodic.

A natural question is the following. Given some $T > 0$, let

$$\mathcal{O}^T_\alpha := \bigcup_{0 \leq t \leq T} X^t_\alpha(0) \subset \mathcal{T}^d_\alpha \tag{1}$$

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be a finite piece of orbit starting at the origin. As T goes to infinity, $\mathcal{O}_\alpha^T$ fills the torus $\mathcal{T}_\alpha^d$, hence given any $\delta > 0$, there exists a smallest positive time $T_\alpha(\delta)$ such that $\mathcal{O}_\alpha^{T_\alpha(\delta)}$ is a δ-dense subset of $\mathcal{T}_\alpha^d$ (for a fixed metric on $\mathcal{T}_\alpha^d$ induced by a choice of a norm on F_α). This time $T_\alpha(\delta)$ is usually called the δ-ergodization time. Note that in (1) we chose the initial condition $\theta_0 = 0$; yet it is obvious that choosing a different θ_0 (and consequently a different leaf of the foliation) lead to the same ergodization time. Then the question is to estimate this time $T(\delta)$ as a function of δ.

To do so, let us define the function

$$\Psi_\alpha(Q) := \max\{|k \cdot \alpha|^{-1} \mid k \in \Lambda_\alpha,\ 0 < |k| \leq Q\}, \tag{2}$$

where, if $k = (k_1, \dots, k_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$k \cdot \alpha = k_1\alpha_1 + \cdots + k_n\alpha_n, \quad |k| = \max\{|k_1|, \dots, |k_n|\}.$$

The function Ψ in (2) is defined for $Q \geq Q_\alpha$, where $Q_\alpha \geq 1$ is the length of the shortest non-zero vector in Λ_α, that is

$$Q_\alpha := \inf\{|k| \mid k \in \Lambda_\alpha \setminus \{0\}\}$$

which depends only on the lattice. Another lattice constant is the co-volume $|\Lambda_\alpha|$ of Λ_α, which is the d-dimensional volume of a fundamental domain of $\mathcal{T}_\alpha^d = F_\alpha/\Lambda_\alpha$, and let us write

$$C_\alpha := |\Lambda_\alpha|^2.$$

Without loss of generality, we may assume that the vector α has a component equals to one; if not, just divide α by $|\alpha|$, and changing its sign if necessary, one just needs to consider $T_\alpha(\delta)/|\alpha|$ instead of $T_\alpha(\delta)$.

We can now state our main result.

Theorem 1. *Let $\delta > 0$ such that $\delta \leq d^2((n+2)Q_\alpha)^{-1}$. Then we have the inequality*

$$T_\alpha(\delta) \leq C_{d,\alpha}\Psi\left(2C_{d,\alpha}\delta^{-1}\right), \quad C_{d,\alpha} := d^2 d! C_\alpha.$$

Even though, up to our knowledge, this result haven't been stated and proved in this generality, it is not essentially new; the novelty lies in its proof.

But first let us recall the previous results, which were dealing only with the case $d = n$ (in this case, one has $Q_\alpha = C_\alpha = 1$ and there is no restriction on $0 < \delta \leq 1$). Assuming moreover that α satisfies the following Diophantine condition:

$$|k \cdot \alpha| \geq \gamma|k|^{-\tau}, \quad k \in \mathbb{Z}^n \setminus \{0\}, \quad \gamma > 0, \quad \tau \geq n-1,$$

the above result reads

$$T_\alpha(\delta) \lesssim \delta^{-\tau}.$$

This result, but with the exponent τ replaced by the worse exponent $\tau+n$, was first established in [CG94], where it was used in the problem of instability of Hamiltonian systems close to integrable (Arnold diffusion). This was then slightly improved in [Dum91] to the value $\tau+n/2$; see also [DDG96] where this ergodization time is shown to be closely related to problems in statistical physics. The estimate with the exponent τ was eventually obtained in [BGW98], and then later the more general statement (without assuming α to be Diophantine) was

obtained in [BBB03]. For a survey on the results and applications of this ergodization time, we refer to [Dum99].

All these proofs are based on Fourier analysis, and it is our purpose to offer a new proof, which is geometric and rather simple. First let us observe that there is a trivial case, namely when $d = 1$. In this case, writing $\alpha = \omega$, the vector is in fact rational, that is $q\omega \in \mathbb{Z}^n$ for some minimal integer $q \geq 1$, and obviously $T_\omega(\delta) = q$ for any $0 \leq \delta \leq 1$: in fact, the orbits of the linear flow X^t_ω are all periodic of period q so for $T = q$, one has the equality $\mathcal{O}^T_\omega = \mathcal{T}^1_\omega$. Our proof will essentially reduce the general case to the trivial case: the proposition below shows that in general the linear flow X^t_α can be approximated by d periodic flows $X^t_{\omega_j}$ with periods q_j, $1 \leq j \leq d$, and such that the vectors $q_j\omega_j \in \mathbb{Z}^n$ span the lattice Λ_α. Here's a precise statement.

Proposition 1. *Let $Q \geq (n+2)Q_\alpha$. Then there exist d rational vectors $\omega_1, \dots, \omega_d$ in $\mathbb{Q}^n$, of denominators $q_1, \dots, q_d$, such that:*

(i) for all $1 \leq j \leq d$, $|\alpha - \omega_j| \leq d(q_jQ)^{-1}$ and $1 \leq q_j \leq dd!C_\alpha\Psi(2d!C_\alpha Q)$;

(ii) the integer vectors $q_1\omega_1, \dots, q_d\omega_d$ form a basis for the lattice Λ_α.

This Proposition was proved in [BF13], see Theorem 2.1 and Proposition 2.3. The only ingredient used there is the following well-known transference result in geometry of numbers (see [Cas59] for instance): if $\mathcal{C}$ and Λ are respectively a convex body and a lattice in a Euclidean space of dimension d, and if $\mathcal{C}^*$ and Λ^* denote their dual, then

$$1 \leq \lambda_k(\mathcal{C}, \Lambda)\lambda_{d+1-k}(\mathcal{C}^*, \Lambda^*) \leq d!, \quad 1 \leq k \leq d$$

where $\lambda_k(\mathcal{C}, \Lambda)$ is the k-th successive minima of $\mathcal{C}$ with respect to Λ.

The proof of Theorem 1 is now a trivial matter if one uses Proposition 1.

Proof of Theorem 1. Choose $Q = d^2\delta^{-1}$. Since we required $\delta \leq d^2((n+2)Q_\alpha)^{-1}$, $Q \geq (n+2)Q_\alpha$ and so Proposition 1 can be applied. It follows from (ii) that the set

$$\{t_1q_1\omega_1 + \cdots + t_dq_d\omega_d \mid (t_1, \dots, t_d) \in [0,1[^d\}$$

is a fundamental domain for $\mathcal{T}^d_\alpha = F_\alpha/\Lambda_\alpha$. Hence given an arbitrary point $\theta^* \in \mathcal{T}^d_\alpha$, there is a unique $(t_1^*, \dots, t_d^*) \in [0,1[^d$ such that

$$\theta^* = t_1^*q_1\omega_1 + \cdots + t_d^*q_d\omega_d.$$

Now by (i), for any $1 \leq j \leq d$, we have

$$|t_j^*q_j\alpha - t_j^*q_j\omega_j| \leq dt_j^*Q^{-1} \leq dQ^{-1}, \quad 1 \leq q_j \leq dd!C_\alpha\Psi(2d!C_\alpha Q), \tag{3}$$

so that if we set $T^* = t_1^*q_1 + \cdots + t_d^*q_d$, the first inequality of (3) gives

$$|T^*\alpha - \theta^*| = \left| \left(\sum_{j=1}^d t_j^*q_j \right) \alpha - \sum_{j=1}^d t_j^*q_j\omega_j \right| \leq \sum_{j=1}^d |t_j^*q_j\alpha - t_j^*q_j\omega_j| \leq d^2Q^{-1} = \delta$$

while the second inequality of (3) gives

$$T^* = \sum_{j=1}^d t_j^*q_j \leq \sum_{j=1}^d q_j \leq d^2d!C_\alpha\Psi(2d!C_\alpha Q) = d^2d!C_\alpha\Psi(2d^2d!C_\alpha\delta^{-1}) = C_{d,\alpha}\Psi(2C_{d,\alpha}\delta^{-1}).$$

The result follows. $\square$

To conclude, let us examine the special case $n = 2$, that is we consider the linear flow associated to $(1, \alpha) \in \mathbb{R}^2$, with $|\alpha| \leq 1$. By the classical processes of suspension and taking section, it is equivalent to consider the circle rotation $R_\alpha : \mathbb{T} \to \mathbb{T}$ given by $R_\alpha(x) = x + \alpha$ mod 1. For any $0 < \delta < 1$, the δ-ergodization time of R_α is the smallest natural number $N = N_\alpha(\delta)$ such that for any $x \in \mathbb{T}$, the finite orbit $\{x, R_\alpha(x), \ldots, R_\alpha^N(x)\}$ is a δ-dense subset of $\mathbb{T}$. Observe that for $\alpha \notin \mathbb{Q}$, $N_\alpha(\delta)$ is always well defined, while for $\alpha = p/q \in \mathbb{Q}^*$, $N_{p/q}(\delta)$ is well-defined if and only if $\delta \geq q^{-1}$ in which case $N_{p/q}(\delta) \leq q - 1$.

Classical proofs in the special case $n = 2$ are usually based on continued fractions (see [DDG96] for instance), and there was a belief that the absence of a good analog of continued fractions in many dimension was an obstacle to extend the known estimate for $n = 2$. We take the opportunity here to give an elementary proof in the case $n = 2$ which does not use continued fractions but simply relies on the Dirichlet's box principle. For simplicity, we shall write $\Psi_\alpha = \Psi_{(1,\alpha)}$ the function defined in (2).

Theorem 2. *If $\alpha \notin \mathbb{Q}$ and $|\alpha| \leq 1$, we have*

$$N_\alpha(\delta) \leq [\Psi_\alpha(2\delta^{-1})] - 1$$

where $[\,.\,]$ denotes the integer part.

Proof. Recall that by Dirichlet's box principle, given any $Q \geq 1$, there exists $(q, p) \in \mathbb{N}^* \times \mathbb{Z}$ relatively prime such that

$$|q\alpha - p| \leq Q^{-1}, \quad 1 \leq q \leq Q.$$

Apply this with $Q = \Psi_\alpha(2\delta^{-1})$: there exists $(q, p) \in \mathbb{N}^* \times \mathbb{Z}$ relatively prime such that

$$|q\alpha - p| \leq \Psi_\alpha(2\delta^{-1})^{-1}, \quad 1 \leq q \leq \Psi_\alpha(2\delta^{-1}).$$

From the second estimate, $q \leq [\Psi_\alpha(2\delta^{-1})]$, so it is enough to show that $N_\alpha(\delta) \leq q - 1$. From the first estimate above, the definition of Ψ_α and the fact that $\max\{|q|, |p|\} = q$ (as $|\alpha| \leq 1$), we have $\Psi_\alpha(q) \geq \Psi_\alpha(2\delta^{-1})$, hence $q \geq 2\delta^{-1}$, $\delta/2 \geq q^{-1}$ and so $N_{p/q}(\delta/2) \leq q - 1$. Using the first estimate again and the fact that $\Psi_\alpha(2\delta^{-1})^{-1} \leq \delta/2$, it is easy to see that the distance between $\{x, R_\alpha(x), \ldots, R_\alpha^{q-1}(x)\}$ and $\{x, R_{p/q}(x), \ldots, R_{p/q}^{q-1}(x)\}$ is at most $\delta/2$. The latter set being $\delta/2$-dense, the former is δ-dense, hence $N_\alpha(\delta) \leq q - 1$ and this ends the proof. $\square$

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